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Self-Consistent Method for Determining the Boundary Shape between a Plasma and a Magnetic Field

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A calculational method of determining the boundary surface between a plasma and a magnetic field is described. The method consists of neglecting the curvature of the surface and approximating the magnetic field adjacent to the boundary by a sum of the field due to the *local* surface current and the plasma-independent magnetic field source. This field is used in the boundary equations to compute the boundary surface. The resulting surface is then used to compute the magnetic field due to the curvature of the surface and the computation of the boundary surface is repeated. Reiteration of the calculational steps is continued until a self-consistent solution is obtained in which the magnetic field resulting from the curvature of the previous surface is used to obtain a surface whose shape does not differ from the previous surface by more than the imprecision of the calculation of the magnetic field. The method is illustrated by application to three simple problems of (1) a line dipole immersed in a plasma exerting a constant pressure on the boundary, (2) a point dipole immersed in plasma exerting constant pressure, and (3) a line dipole in a plasma stream exerting a pressure in only one direction (parallel to the stream velocity).

author

INTRODUCTION

THE physics of the boundary layer between a plasma and a magnetic field has been discussed by several authors,[1-6] who have shown that, if an electric field due to charge separation exists in the boundary layer, the boundary layer will usually be very thin, $3.8 \times 10^5/n_e^{\frac{1}{2}}$ cms, where n_e is the number of electrons/cc in the plasma. If the thickness of the boundary layer is very much less than the radius of curvature of the boundary, due either to the electric field in the boundary layer or a small ion cyclotron radius in the magnetic field, then reflection from a smooth boundary layer will be specular, for the electric field perpendicular to the boundary and the magnetic field parallel to the boundary will be the same for the exit path of the particles as for their entrance path. In this case the particle pressure of the plasma on the surface is easily determined.

The shape of the boundary surface is such that the electrical currents in the boundary layer cause the magnetic field (due to surface currents and plasma-independent sources of field) to be zero in the plasma and the magnetic pressure adjacent but exterior to the plasma to be equal to the particle pressure on the boundary. Unfortunately the boundary shape cannot be calculated, in general, from these boundary conditions until the field adjacent to the boundary is known, and the field adjacent to the boundary cannot be determined until the shape of the boundary is known. A calculational method of surmounting this circular difficulty is to neglect all fields due to the surface currents except the field due to the surface current at the point where the surface is being calculated (the *local* surface-current field). This reduces the statement of the problem of the surface calculation to a differential equation. This first approximation yields a very good approximate boundary shape in those regions where the surface current does not vary greatly in direction or magnitude as one moves a significant fraction of the curvature radius along the surface. Near a surface singularity such as exists over a magnetic dipole, the first approximation is very poor. In any case, however, the differential equation may be solved in order to yield a first approximation surface as a second link in a reiterative procedure which ultimately converges to a self-consistent field and surface. The *complete* magnetic field resulting from the currents on this surface may be computed numerically and then be used to compute a second surface. A surface is thus rapidly obtained which

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[1] V. C. A. Ferraro, J. Geophys. Res. 57, 15 (1952).
[2] J. Dungey, *Cosmic Electrodynamics* (Cambridge University Press, Cambridge, England, 1958).
[3] M. N. Rosenbluth, "Dynamics of a Pinched Gas" in *Magnetohydrodynamics*, edited by R. Landshoff (Stanford University Press, Palo Alto, California, 1957).
[4] W. Paskievici, A. Sestero, and H. Weitzner, Courant Institute of Mathematical Sciences, New York University, Rept. NYO 9193 (1962).
[5] J. Hurley, Phys. Fluids 6, 83 (1963).
[6] D. B. Beard, J. Geophys. Res. 65, 3559 (1960).

yields fields which do not differ from the previous surface and fields by more than the numerical imprecision with which the fields are computed.

Some questions concerning the reliability of the first approximation of this method have legitimately been raised recently by Midgley and Davis[7] who used an especially severe example for testing the method in that the problem they chose contained a bad singularity. On the other hand, Hurley[8] and then Spreiter and Briggs[9] have illustrated the excellence of the first approximation by comparing the approximate result to the exact solution when applied to a more tractable problem. Although the first approximation[6] and in a crude fragmentary way the second approximation[10] have been described before when applied to a point dipole in a steady zero-temperature plasma wind, a complete self-consistent solution of any problem has never been given. In the following we apply it to three elementary problems which have been solved previously by other means to illustrate the method and to demonstrate its effectiveness and rapid convergence.

FORMAL STATEMENT OF THE SELF-CONSISTENT METHOD

We seek a surface whose equation in spherical coordinates is $F(r, \theta, \phi)$ = constant. Let the constant be zero and let

$$F(r, \theta, \phi) = r - R(\theta, \phi) = 0. \tag{1}$$

The unit normal vector to the surface, $\hat{\mathbf{n}}_s$, is given by

$$\hat{\mathbf{n}}_s = \nabla F / |\nabla F|, \tag{2}$$

$$\hat{\mathbf{n}}_s = \frac{\left[\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial R}{\partial \theta}\hat{\boldsymbol{\theta}} - \frac{1}{r \sin\theta}\frac{\partial R}{\partial \phi}\hat{\boldsymbol{\phi}}\right]}{\left[1 + \frac{1}{r^2}\left(\frac{\partial R}{\partial \theta}\right)^2 + \frac{1}{r^2 \sin^2\theta}\left(\frac{\partial R}{\partial \phi}\right)^2\right]^{\frac{1}{2}}}. \tag{2a}$$

The boundary conditions in terms of the magnetic field $\mathbf{B}$, adjacent to the boundary and exterior to the plasma are

$$\hat{\mathbf{n}}_s \cdot \mathbf{B} = 0 \tag{3}$$

and

$$|\hat{\mathbf{n}}_s \times \mathbf{B}|^2 = 2\,\mu_0 p, \tag{4}$$

where p is the particle pressure of the plasma on the boundary and the units are mks units. A further occasionally useful boundary condition is that there is no magnetic field in the plasma. The first approximation consists of letting

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_p, \tag{5}$$

where $\mathbf{B}_0$ is the magnetic field due to any source other than the surface currents on the boundary and $\mathbf{B}_p$ is the magnetic field due to a surface current in a plane. It is the magnetic field due to the *local* surface current and would be exactly correct if the surface curvature were infinite. Since the magnetic field is equal in magnitude and opposite in direction on the opposite sides of a current in a plane, in the plasma where the total magnetic field is zero, $\mathbf{B}_p = -\hat{\mathbf{n}}_s \times \mathbf{B}_0$, and hence outside the plasma

$$\hat{\mathbf{n}}_s \times \mathbf{B} = 2\,\hat{\mathbf{n}}_s \times \mathbf{B}_0. \tag{6}$$

Maxwell's equations are, of course, not satisfied in this first approximation. The surface currents may not even be conserved, but the surface shape, aside from singularities, is a good one. The equations do, however, become satisfied in higher approximation.

When Eqs. (2) and (6) are substituted in Eq. (4) using a given field configuration $\mathbf{B}_0$ and a given particle pressure p, the resulting differential equation is solved analytically or numerically, and a surface is obtained. The field due to the curvature of this surface is given by

$$\mathbf{B}_c = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J} \times \mathbf{r}'}{r'^3}\, ds, \tag{7}$$

where $\mathbf{r}'$ is the vector from a point on the surface at which $\mathbf{B}_c$ is evaluated to the differential surface element ds bearing a surface current $\mathbf{J}$. The change in the tangential component of the field in crossing a current sheath is given by $\hat{\mathbf{n}}_s \times \mathbf{B} = \mu_0 \mathbf{J}$ where $\mathbf{J}$ is the sheath current per unit length perpendicular to the current vector and the change in magnetic field is $\mathbf{B}$ for our work since the field in the plasma is assumed zero. Therefore the current magnitude is given in terms of the pressure by

$$J = |\mathbf{n}_s \times \mathbf{B}_0| / \mu_0 = (2p/\mu_0)^{\frac{1}{2}}. \tag{8}$$

Since Eq. (7) represents a correction field due to the curvature of the surface, the point at which $\mathbf{B}_c$ is evaluated must be at the center of the surface layer (not infinitesimally to one side of it since then $\mathbf{B}_c$ would include the field from the local surface current, the planar field). One effective method we used in evaluating this integral, which also helped to smooth out the effect of any spurious surface ripple, was to evaluate $\mathbf{B}_c$ at two pairs of points on each side of the surface, to average the field of each equidistant pair and then to extrapolate the two results to the center of the surface. That is, evaluate the field at $\mathbf{r} + \mathbf{s}_1$, $\mathbf{r} - \mathbf{s}_1$, $\mathbf{r} + \mathbf{s}_2$, and

[7] J. Midgley and L. Davis, Jr., J. Geophys. Res. 67, 499 (1962).
[8] J. Hurley, Phys. Fluids 4, 854 (1961).
[9] J. A. Spreiter and B. R. Briggs, J. Geophys. Res. 67, 37 (1962).
[10] D. B. Beard, J. Geophys. Res. 67, 477 (1962).

$\mathbf{r} - \mathbf{s}_2$, where $\mathbf{s}_1$ and $\mathbf{s}_2$ are small displacements perpendicular to the surface, then

$$\mathbf{B}_c(\mathbf{r}) = \tfrac{1}{2}[\mathbf{B}_c(\mathbf{r} + \mathbf{s}_1) + \mathbf{B}_c(\mathbf{r} - \mathbf{s}_1)] - \tfrac{1}{2}[\mathbf{B}_c(\mathbf{r} + \mathbf{s}_2) + \mathbf{B}_c(\mathbf{r} - \mathbf{s}_2) - \mathbf{B}_c(\mathbf{r} + \mathbf{s}_1) - \mathbf{B}_c(\mathbf{r} - \mathbf{s}_1)]s_1/(s_2 - s_1). \quad (9)$$

In the second and higher approximations, Eq. (5) is replaced by

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_p + \mathbf{B}_c. \quad (10)$$

Since $\mathbf{B}_p$ outside the surface is equal in magnitude and opposite in direction to $\mathbf{B}_p$ inside the surface and $\mathbf{B}$ outside the surface is zero, $\mathbf{B}_p$ inside the surface is

$$\mathbf{B}_p = \mathbf{B}_0 + \mathbf{B}_c.$$

Therefore

$$\mathbf{B} = 2(\mathbf{B}_0 + \mathbf{B}_c)$$

and Eq. (4) yields the full equation

$$|\mathbf{n}_s \times (\mathbf{B}_0 + \mathbf{B}_c)| = \pm(\mu_0/2)^{\frac{1}{2}}p^{\frac{1}{2}}. \quad (11)$$

A LINE DIPOLE IN A HOT AMBIENT PLASMA

The problem of a line dipole surrounded by a hot plasma exerting a constant pressure everywhere on the boundary of the magnetic field has been solved exactly by Cole and Huth[11] using a conformal mapping technique. We use this simple two-dimensional example as a particularly exacting test of the self-consistent method since it has a singularity over the pole. In two dimensional polar coordinates the magnetic field from the plasma-independent source (supposedly two infinite antiparallel current bearing wires) is written

$$\mathbf{B}_0 = (M/r^2)(\sin\theta\,\hat{\mathbf{r}} - \cos\theta\,\hat{\boldsymbol{\theta}}), \quad (12)$$

where M is the two-dimensional dipole moment strength and θ is measured from the magnetic equator.

$$\hat{\mathbf{n}}_s = \left[\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial R}{\partial\theta}\hat{\boldsymbol{\theta}}\right]\left[1 + \frac{1}{r^2}\left(\frac{\partial R}{\partial\theta}\right)^2\right]^{-\frac{1}{2}} \quad (13)$$

and the curvature field, Eq. (7), is obtained from

$$\mathbf{B}_c(\theta_0) = \frac{B_0'}{2\pi}\left\{\int_{-\pi/2}^{\pi/2}\left[\frac{r'\sin(\theta-\theta_0)(r'^2+(dR'/d\theta)^2)^{\frac{1}{2}}\,d\theta}{r'^2+r^2-2r'r\cos(\theta-\theta_0)}\hat{\mathbf{r}} + \frac{[r-r'\cos(\theta-\theta_0)][r'^2+(dR'/d\theta)^2]^{\frac{1}{2}}\,d\theta}{r'^2+r^2-2r'r\cos(\theta-\theta_0)}\hat{\boldsymbol{\theta}}\right] - \int_{\pi/2}^{3\pi/2}\left[\frac{r'\sin(\theta-\theta_0)[r'^2+(dR'/d\theta)^2]\,d\theta}{r'^2+r^2-2r'r\cos(\theta-\theta_0)}\hat{\mathbf{r}} + \frac{[r-r'\cos(\theta-\theta_0)][r'^2+(dR'/d\theta)^2]^{\frac{1}{2}}\,d\theta}{r'^2+r^2-2r'r\cos(\theta-\theta_0)}\hat{\boldsymbol{\theta}}\right]\right\}, \quad (14)$$

[11] J. D. Cole and J. H. Huth, Phys. Fluids **2**, 624 (1959).

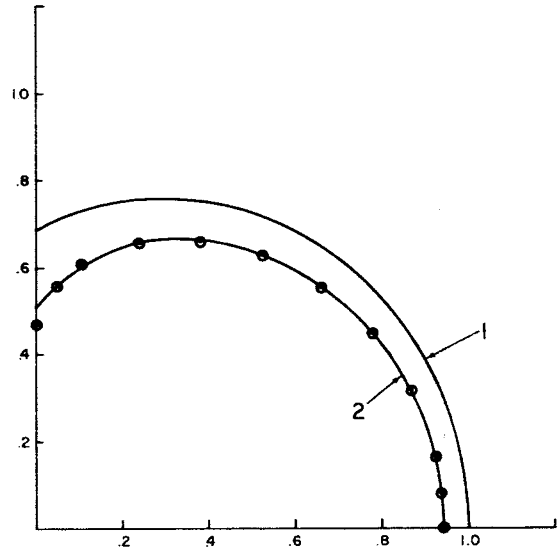

FIG. 1. A line dipole in a hot plasma. The first and second iterated plasma–field boundary surfaces of the standard self-consistent approximation ($|\mathbf{n}_s \times \mathbf{B}|^2 = 2p\mu_0$) appropriately labeled. The exact solution due to Cole and Huth[11] is indicated by the open circles.

where B_0' is $(\mu_0 p/2)^{1/2}$ by use of Eq. (8). At the magnetic equator, Eqs. (6) and (4) yield a convenient unit of length, r_0, defined by

$$r_0 = (2M^2/\mu_0 p)^{\frac{1}{4}}. \quad (15)$$

This is the height of the boundary surface at the magnetic equator in first approximation.

Any one of the three boundary conditions may be used to obtain a self-consistent solution. In all cases, however, the most satisfactory boundary condition from a convergence standpoint is obtained with Eq. (11). We will refer to this as the *standard* self-consistent approximation

$$\frac{\{r(B_{c\theta} - \cos\theta/r^2) + (dR/d\theta)(B_{cr} + \sin\theta/r^2)\}}{[r^2 + (dR/d\theta)^2]^{\frac{1}{2}}} = 1, \quad (16)$$

$$\frac{dR}{d\theta} = r\,\frac{(\sin\theta + r^2B_{cr})(\cos\theta - r^2B_{c\theta})}{(\sin\theta + r^2B_{cr})^2 - r^4} - \frac{r^2[(\sin\theta + r^2B_{cr})^2 + (\cos\theta - r^2B_{c\theta})^2 - r^4]^{\frac{1}{2}}}{(\sin\theta + r^2B_{cr})^2 - r^4}, \quad (17)$$

where the subscripts cr and $c\theta$ refer to the components of the curvature field and all distances are measured in units of r_0. One difficulty which invariably arises with this boundary condition is that the argument of the square root appearing in Eq. (17) is the square of the total magnetic field outside

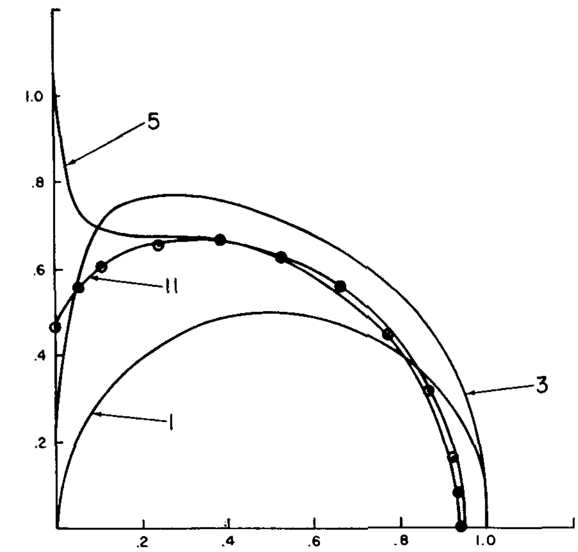

FIG. 2. A line dipole in a hot plasma. The first, second, and third iterated plasma–field boundary surfaces based on the boundary condition that the field near the boundary layer is a constant. The exact solution is indicated by open circles.

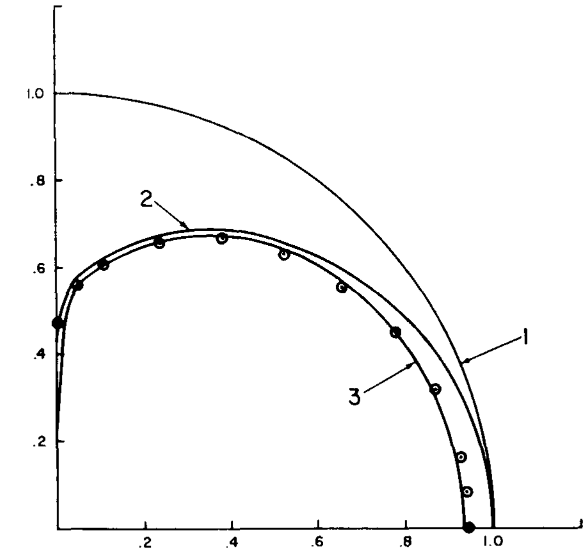

FIG. 3. A line dipole in a hot plasma. The first, third, fifth, and eleventh iterated plasma–field boundary surfaces based on the boundary condition that $\hat{\mathbf{n}}_s \cdot \mathbf{B} = 0$. The exact solution is indicated by open circles.

the surface which in the ultimate solution is zero. The approximate magnetic fields used in this argument can add up to a total field less than zero and thus no real solution to the surface equation exists for some approximate magnetic fields. Either an arbitrary surface must be invented in the region where no solution exists or a new calculational procedure must be used. One simple and successful procedure, which we frequently adopted, was to take the square root of the absolute value of the argument of the square root and change the sign of the square root when the argument was negative. The first, second, and third surfaces resulting from Eq. (17) and the use of an IBM 7094 machine computer are compared to the exact solution in Fig. 1. The problem can be iterated on a machine for as many times as desired and many surfaces obtained all within the error made in computing the curvature magnetic field. The choice of which surface to take and when to stop the iteration can be decided by the Midgley and Davis criterion of surface accuracy, namely computing the ratio of the field in the plasma (which should be zero) to the dipole field alone.

The precision of the final result, independent of the boundary condition used to obtain it, depends on the precision with which the surface curvature field (the nonlocal field) is computed. The size of the interval used in the numerical integration and the trouble taken with the singularity occurring in the surface integration must be chosen as a compromise between machine time and desired accuracy. The first approximate surface is so easy to obtain that it can invariably be obtained without the use of a machine computer. The machine time required by the higher order approximations is taken up with the surface integration required for computing B_c. Approximately two and a half minutes was required for each iteration if one degree intervals were used. We have found one degree integration intervals result in a surface error of no more than two percent in these simple test problems. Other methods[7,12] of finding the boundary surface face the same problem of computing the surface fields with an optimum choice of integration interval for least machine time for the precision desired.

Two other boundary conditions are possible which may also be used to give a self-consistent solution. (i) The magnetic field is a constant adjacent to the plasma so that

$$\left(\frac{1}{r^2}\sin\theta + B_{cr}\right)^2 + \left(\frac{-1}{r^2}\cos\theta + B_{c\theta}\right)^2 = 1. \qquad (18)$$

The first, second, and third surfaces resulting from this algebraic equation and the use of an IBM 7094 machine computer are compared to the exact solution in Fig. 2. (ii) Use of Eq. (3) for this problem leads to the simple differential equation

$$dR/d\theta = r[(\sin\theta + r^2B_{cr})/(\cos\theta - r^2B_{c\theta})]. \qquad (19)$$

[12] R. J. Slutz, J. Geophys. Res. **67**, 505 (1962).

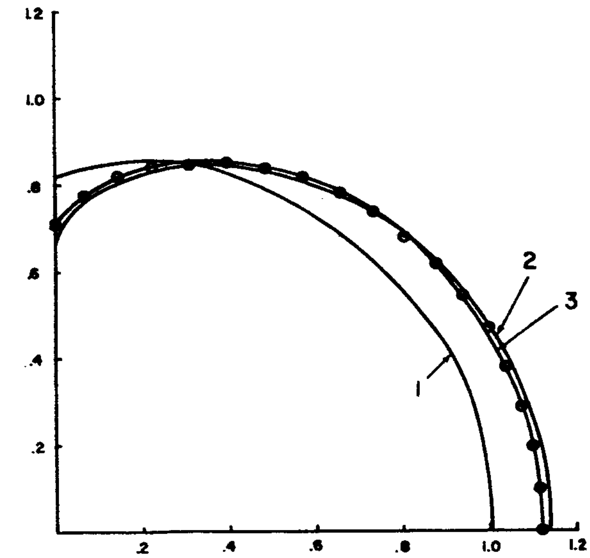

FIG. 4. A point dipole in a hot plasma. The first, second, and third iterated plasma–field boundary surfaces of the standard self-consistent approximation ($|\hat{n}_s \times \mathbf{B}|^2 = 2p\mu_0$). The solution due to Midgley and Davis[7] is indicated by open circles.

The first, third, fifth, and eleventh surfaces resulting from Eq. (19) and the use of an IBM 7094 machine computer are compared to the exact solution in Fig. 3. This boundary condition results in an especially slow convergence of the iterative process.

A POINT DIPOLE IN A HOT AMBIENT PLASMA

The problem of a point dipole surrounded by a hot plasma exerting a constant pressure everywhere on the boundary of the magnetic field has been solved approximately by Midgley and Davis[7] and Slutz.[12] Midgley and Davis[7] have shown that their solution obtained by a moment technique gives a magnetic field in the plasma of about 10^{-5} of the dipole field except near the magnetic pole. (Because of the difficulty in magnetic field computation their excellent ratios are not computed absolutely but in comparison to a sphere with the current proportional to sin θ which enables them for comparison purposes to approximately calculate a dipole field by means of the same field-calculational method.) Their surface is thus a good comparison surface except within 5° of the pole where their surface fails to approach the polar axis with sufficiently steep slope. (An infinite slope at the axis is required from theoretical considerations.) Midgley and Davis chose this problem partly as a test of the first approximation to the self-consistent method and found that near the magnetic pole the first standard self-consistent approximation is rather poor as indeed it seems to be near any singularity (cusp) in a boundary surface. We choose this problem in order to examine how the method fares in higher approximation.

The dipole magnetic field in polar coordinates is given by

$$\mathbf{B}_0 = (-M/r^3)(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}) \tag{20}$$

The unit surface normal vector has been given in Eq. (2a). Due to symmetry, however, there is no dependence on ϕ. The curvature field, Eq. (7), is obtained from

$$\mathbf{B}_c = \frac{B_0'}{\pi}\int_0^{\pi} \frac{[r'^2 + (dR'/d\theta)^2]^{\frac{1}{2}}}{[(a+x)^2+z^2]^{\frac{1}{2}}} \cdot \left\{ \frac{z}{\rho}\left[-K(k) + \frac{a^2+x^2+z^2}{(a-x)^2+z^2}E(k)\right]\hat{\mathbf{x}} + \left[K(k) + \frac{a^2-x^2-z^2}{(a-x)^2+z^2}E(k)\right]\hat{\mathbf{z}}\right\} d\theta, \tag{21}$$

where $K(k)$ and $E(k)$ are elliptic integrals of the first and second kind, and

$$z = r\cos\theta_0 - r'\cos\theta, \quad a = r'\sin\theta, \quad x = r\sin\theta_0,$$
$$k = (4ax)^{\frac{1}{2}}[(a+x)^2+z^2]^{-\frac{1}{2}},$$

where $r'(\theta)$ is a dependent variable and $r(\theta_0)$ is the point at which $\mathbf{B}_c$ is computed. The height of the boundary surface at the magnetic equator in first approximation is used as the length unit in which all distances are measured:

$$r_0 = (2M^2/\mu_0 p)^{\frac{1}{4}}. \tag{22}$$

Equation (11) for this problem becomes

$$\frac{\{r(B_{c\theta}+\sin\theta/r^3)+(dR/d\theta)(B_{cr}+2\cos\theta/r^3)\}}{[r^2+(dR/d\theta)^2]^{\frac{1}{2}}} = 1 \tag{23}$$

$$\frac{dR}{d\theta} = r\,\frac{-(2\cos\theta + r^3B_{cr})(\sin\theta + r^3B_{c\theta})}{(2\cos\theta + r^3B_{cr})^2 - r^6} + \frac{r^3[(\sin\theta + r^3B_{c\theta})^2 + (2\cos\theta + r^3B_{cr})^2 - r^6]^{\frac{1}{2}}}{(2\cos\theta + r^3B_{cr})^2 - r^6}. \tag{24}$$

The behavior and physical significance of the square root in Eq. (24) has been commented on previously following Eq. (17). The first, second, and third surfaces resulting from Eq. (24) are compared with the Midgley and Davis and Slutz solution in Fig. 4. The magnetic field was computed in the plasma at various points as Midgley and Davis did in testing their solution. The third surface gave a ratio of the field in the plasma to the dipole field which is less than or equal to the estimated imprecision in the surface curvature field computation everywhere. Thus in three short steps we were able to converge to a self-consistent surface for this particularly difficult test of the technique.

The null component of **B** perpendicular to the boundary can also be used to obtain a solution. Use of Eq. (3) for this problem leads to the differential equation

$$dR/d\theta = r[(2\cos\theta + r^3 B_{cr})/(\sin\theta + r^3 B_{c\theta})]. \qquad (25)$$

The first, twelfth, twenty-first, and forty-sixth surfaces resulting from Eq. (25) are compared to the Slutz and Midgley and Davis solution in Fig. 5.

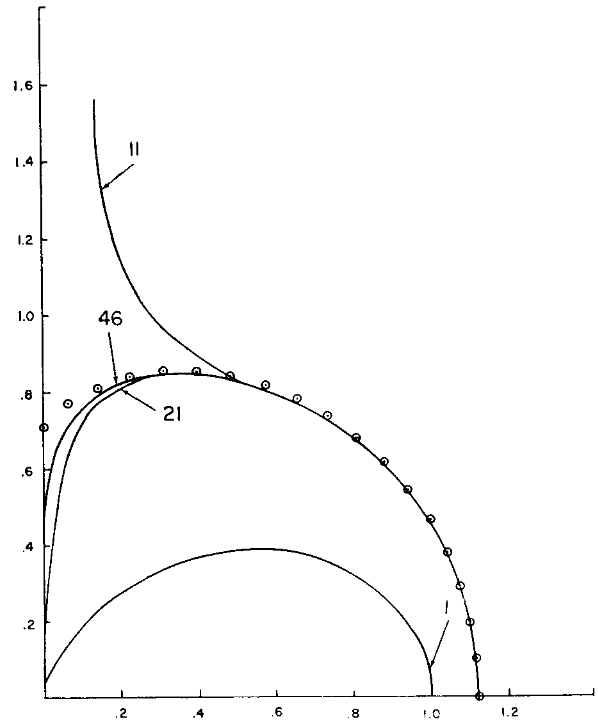

FIG. 5. A point dipole in a hot plasma. The first, twelfth, twenty-first, and forty-sixth iterated plasma–field boundary surfaces based on the boundary condition that $\hat{\mathbf{n}}_s \cdot \mathbf{B} = 0$. Midgley and Davis'[7] solution is indicated by open circles.

A LINE DIPOLE IN A COLD PLASMA WIND

The problem of a line dipole upon which a cold plasma blows, whose ions and electrons all have identical velocity perpendicular to the dipole, has been solved exactly by Zhigulev and Romishevskii,[13] Hurley,[8] and Dungey.[14] It more closely resembles the original problem to which the self-consistent method was applied[6] (the geomagnetic dipole in the solar wind) than the other problems do. Because of the great interest of the geophysical problem, the line dipole is of exceptional interest as a test of the convergence of the method.

Since the particles are specularly reflected at the boundary layer, the pressure on the boundary is given by

$$p = (nv\cos\psi)(2mv\cos\psi)$$
$$= 2nmv^2\cos^2\psi = p_0\cos^2\psi, \qquad (26)$$

where the first expression in parentheses is the number of ions striking the surface per unit area, the second is the change in momentum of the ions, and ψ is the angle between the normal to the surface and the wind velocity vector. The velocity vector of the wind is

$$\mathbf{v} = v(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}).$$

Therefore

$$p = p_0(\hat{\mathbf{n}}_s \cdot \hat{\mathbf{v}})^2$$
$$= p_0\left(\cos\theta + \frac{1}{r}\frac{dR}{d\theta}\sin\theta\right)^2 \Big/ \left[1 + \frac{1}{r^2}\left(\frac{dR}{d\theta}\right)^2\right]. \qquad (27)$$

When Eqs. (12) and (27) are substituted into Eq. (11) we obtain

$$\frac{[r(B_{c\theta} - \cos\theta/r^2) + (dR/d\theta)(C_{cr} + \sin\theta/r^2)]}{[r^2 + (dR/d\theta)^2]^{\frac{1}{2}}}$$
$$= \left|r\cos\theta + \frac{dR}{d\theta}\sin\theta\right| \Big/ \left[r^2 + \left(\frac{dR}{d\theta}\right)^2\right]^{\frac{1}{2}}, \qquad (28)$$

$$\frac{dR}{d\theta} = r\cot\theta\,\frac{1 - r^2(1 + \sec\theta\,B_{c\theta})}{1 + r^2(1 + \csc\theta\,B_{cr})}. \qquad (29)$$

The sign in the square root is taken so that the right-hand side of Eq. (11) will be positive; since $\hat{\mathbf{n}}_s \cdot \hat{\mathbf{v}}$ must always be negative in order for the external surface of the plasma to always be presented to the wind, the negative sign of the square root must be taken. At the neutral point, the point where $\mathbf{v}$ is parallel to the surface and $\hat{\mathbf{n}}_s \cdot \hat{\mathbf{v}} = 0$, **B** and the surface current change direction so that the left-hand side of Eq. (28) must be multiplied by a minus sign to keep it positive. Hence beyond the null point the surface equation becomes

$$\frac{dR}{d\theta} = r\cot\theta\,\frac{1 - r^2(\sec\theta\,B_{c\theta} - 1)}{1 + r^2(\csc\theta\,B_{cr} - 1)}. \qquad (30)$$

The curvature field is given by

$$\mathbf{B}_c = \frac{B_0'}{2\pi}\int_{-\theta_n}^{\theta_n} \frac{[r'\cos\theta + (dR'/d\theta)\sin\theta]}{r'^2 + r^2 - 2rr'\cos(\theta - \theta_0)}$$
$$\cdot \{r'\sin(\theta - \theta_0)\hat{\mathbf{r}} + [r - r'\sin(\theta - \theta_0)]\hat{\boldsymbol{\theta}}\}\,d\theta$$
$$- \frac{B_0'}{2\pi}\int_{\theta_n}^{2\pi-\theta_n} \frac{[r'\cos\theta + (dR'/d\theta)\sin\theta]}{r'^2 + r^2 - 2rr'\cos(\theta - \theta_0)}$$
$$\cdot \{r'\sin(\theta - \theta_0)\hat{\mathbf{r}} + [r - r'\sin(\theta - \theta_0)]\hat{\boldsymbol{\theta}}\}\,d\theta, \qquad (31)$$

[13] V. N. Zhigulev and E. A. Romishevskii, Dokl. Akad. Nauk SSSR **127**, 1001 (1959) [English transl.: Soviet Phys.—Doklady **4**, 859 (1959)].

[14] J. Dungey, J. Geophys. Res. **66**, 1043 (1961).

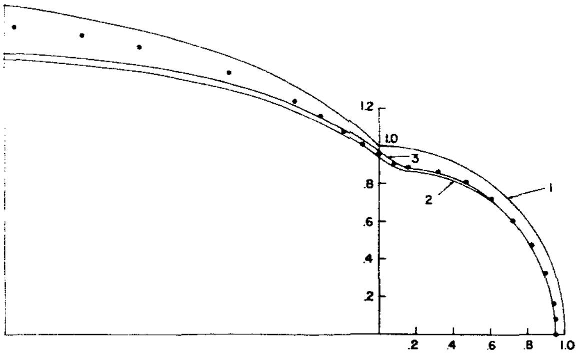

FIG. 6. A line dipole in a cold plasma wind. The first, second, and third iterated plasma-boundary surfaces of the standard self-consistent approximation. The exact solution is indicated by open circles.

where θ_n is the position of the neutral point. The resulting first, second, and third surfaces are compared to Hurley's graph of his exact solution in Fig. 6. Note that the first approximation is somewhat arbitrary in that the solution on the lee side of the line dipole is simply that solution of Eq. (31) which intersects the windward surface at the pole (the neutral point of the first approximation); the discontinuity in the derivative causes an infinite curvature field perpendicular to the surface at this point but it smooths out in higher approximation.

CONCLUSIONS

A method of determining the shape of the boundary between a plasma and a magnetic field has been described and illustrated by application to the three simple cases of a line dipole with a constant plasma pressure everywhere on the boundary, a line dipole with a variable plasma pressure on the boundary which depends on the direction of the surface normal, and a point dipole subject to a constant plasma pressure on the boundary. The method consists of first calculating the surface using an approximate magnetic field which ignores the curvature of the surface, then using the magnetic field computed from the currents flowing on this first surface and so on until no significant change results between two consecutively calculated surfaces. We have called this the self-consistent method since the magnetic field computed from the last calculated surface will yield the same surface if it is used to calculate the surface again.

The most difficult and machine-time consuming part of the calculation turned out to be computing the magnetic fields. The self-consistent method in every case quickly converged to the final result by the third computed surface. The precision of the final result was always limited by the precision with which the magnetic fields were computed.

As a final test of the stability of the method, the exact surface or best previously computed answer (Midgley and Davis for the point dipole) was used as a starting surface and the problem was run on the machine for ten or more iterations. The final surface (except for a better polar solution for the point dipole) differed from the starting surface in this instance by less than the imprecision in computing the fields. This agreement gave us some confidence that when the correct magnetic fields are used, the method will not lead to an instability in which succeeding surfaces will start to wander from the true surface giving more and more misleading magnetic fields.

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